OPTIMUM PREVENTATIVE SAMPLING BY THE DISCRETE MAXIMUM PRINCIPLE

by

SUBIR BANERJI

B. S. (Mathematics) University of Calcutta
Calcutta, India, 1959

B. S. (Mechanical Engineering) Government College of
Engineering & Technology, Raipur, India, 1964

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Approved by:

Ching-Lai Hwang
Major Professor

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1. INTRODUCTION

Extensive work has been done over the past decade on the application of the maximum principle to optimal control problems with encouraging and rewarding results. Originally the maximum principle was developed by the Russian mathematician Pontryagin in 1956 for the continuous process and was applied mainly in the field of optimum system control. The discrete version of the maximum principle is an analogy of the continuous process and was developed by Rozoner, Chang, Katz and Fan and Wang (3). The principle has recently been applied to a variety of industrial management problems like production scheduling (5), transportation problem (6), system reliability (4), equipment replacement (1), traffic flow (10), design of gear-train (2) and so on. However, not much work has been done on quality control which is also a vital organ in an organization.

The aim of this report is to show the applicability of the discrete maximum principle in the field of quality control.

A brief review of the discrete maximum principle and the recurrence relation for one-dimensional process is presented in section 2. A more detailed analysis may be found in the works of Fan and Wang (3).

In section 3 "Ordinary Sampling" has first been explained and then optimum decision criteria for an N-stage process

have been developed with the help of the discrete maximum principle. A numerical example for a 3-stage system is then solved.

Section 4 deals with preventative sampling (9). The central idea of sampling is not only to find faults but also to prevent their future occurence. It is rightly said that "Quality control is the science of preventing the manufacture of defective product." Juran (7) says "There must be a recognition of the fact that the basic objective is prevention of defects and that all else is secondary." All these ideas are in recognition of the principle that it is better to prevent defects from happening than to let them happen and then to make the best of it.

There are a number of preventative measures like process control capabilities, control chart analysis, preventative sampling, design of sampling plans etc. For this report we are considering preventative measures due to human reactions and other intangible factors. The effect of human reactions on sampling can be realized from the ineffectiveness of the sampling plan in which no provision is made for notifying the producing operator of the rejection of a lot. Incorporating a slight modification of notifying the operator about the rejection of a lot has found to have a positive effect in reducing future defects.

As for practical examples we can think of the random sampling for tax return, checking the drivers license of a teen-ager or a conductor checking a ticket in a bus. One

hundred percent checking in these situations is cost prohibitive whereas no inspection will lead to abuse of the law. Hence random sampling is the only answer. The knowledge that sampling is being done deters people from infringing the law.

Three types of optimum preventative problems are encountered in practice. They are

1. Minimizing total expected cost--the total sampling volume being given,
2. Minimizing total expected cost--the cost of inspection being considered,
3. Minimizing total expected cost--the cost of inspection and the total sampling volume being given.

Section 5 is devoted to these types of problems. A general solution for N-stages is worked out and then a numerical example for each type of problem is solved by making use of the general solution.

2. REVIEW OF A DISCRETE FORM OF THE MAXIMUM PRINCIPLE

A multistage system with N stages in series is shown in Fig. 1. The process consists of N stages connected in series. The state of the process stream denoted by an s-dimensional vector, $x = (x_1, x_2, \ldots, x_s)$, is transformed at each stage according to an r-dimensional decision vector, $\theta = (\theta_1, \theta_2, \ldots, \theta_r)$, which represents the decision made at that stage. The transformation equation at the nth stage may be written, in vector form, as follows

$$x^n = T^n(x^{n-1}; \theta^n), \quad n = 1, 2, \ldots, N, \qquad (2\text{-}1)$$
$$x^0 = \alpha .$$

The optimization problem associated with such a system is to find a sequence of decision variables θ^n, $n = 1, 2, \ldots, N$, subject to constraints

$$\Psi_i^n(\theta_1^n, \theta_2^n, \ldots, \theta_r^n) \leq 0, \quad n = 1, 2, \ldots, N, \quad i = 1, 2, \ldots, r, \qquad (2\text{-}2)$$

which makes a function of final state variables

$$S = \sum_{i=1}^{s} c_i x_i^N, \qquad c_i = \text{constant}, \qquad (2\text{-}3)$$

an extremum when the initial condition $x^0 = \alpha$ is given.

The procedure for solving such an optimization problem by a discrete version of the maximum principle is to

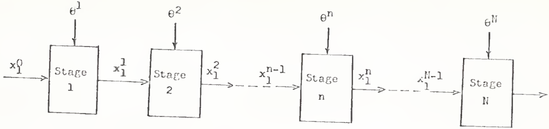

Fig. 1. Multistage decision process

introduce an s dimensional adjoint vector z^n and a Hamiltonian function H^n which satisfy the following relations

$$H^n = \sum_{i=1}^{s} z_i^n T_i^n (x^{n-1}; \theta^n), \qquad n = 1, 2, \ldots, N, \qquad (2\text{-}4)$$

$$z_i^{n-1} = \frac{\partial H^n}{\partial x_i^{n-1}}, \qquad n = 1, 2, \ldots, N; \quad i = 1, 2, \ldots, s, \qquad (2\text{-}5)$$

and

$$z_i^N = c_i, \qquad i = 1, 2, \ldots, s. \qquad (2\text{-}6)$$

If the optimal decision vector function, θ^n, which makes the objective function S an optimum, is interior to the set of admissible decisions, θ^n, (the set given by equation (2-2)), a necessary condition for S to be an (local) extremum with respect to θ^n is

$$\frac{\partial H^n}{\partial \theta^n} = 0, \qquad n = 1, 2, 3, \ldots, N. \qquad (2\text{-}7)$$

If θ^n is at a boundary of the set, it can be determined from the condition that H^n is (locally) an extremum.

This is the basic algorithm of the discrete maximum principle. In the above formulation of the problem we see that only initial conditions of the state variables are fixed, but in practice many cases might arise where the final conditions of the state variable might also be fixed. For

example the final conditions x_a^N and x_b^N may be preassigned and the objective function given as

$$S = \sum_{\substack{i=1 \\ i \neq a \\ i \neq b}}^{s} c_i x_i^N . \tag{2-8}$$

Under such conditions the basic algorithm is still applicable, except that equation (2-6) is changed to

$$z_i^N = c_i \,, \qquad i = 1, 2, \ldots, s, \quad i \neq a, b. \tag{2-9}$$

Optimization of One Dimensional Processes (3)

If a multistage decision process can be completely characterized for the purpose of optimization by a single state variable, the process is called a one-dimensional multistage decision process.

For one dimensional process, there is only one state variable x_1, satisfying the performance equation

$$x_1^n = T^n(x_1^{n-1}; \theta^n), \qquad n = 1, 2, \ldots, N. \tag{2-10}$$

In general, the objective function to be optimized is the sum of a certain function of x_1 and θ over all stages of the system such as

$$\sum_{n=1}^{N} G(x_1^{n-1}; \theta^n) \; .$$

The optimization problem associated with such a process is to find a sequence of decision variables θ^n, $n = 1, 2, \ldots, N$ so as to maximize

$$\sum_{n=1}^{N} G(x_1^{n-1};\ \theta^n),$$

when x_1^0 is given. Introducing a new state variable x_2^n satisfying

$$x_2^n = x_2^{n-1} + G(x_1^{n-1};\ \theta^n), \quad n = 1, 2, \ldots, N, \tag{2-11}$$
$$x_2^0 = 0.$$

Therefore, we see that the objective function is given by

$$S = \sum_{n=1}^{N} G(x_1^{n-1};\ \theta^n) = x_2^N . \tag{2-12}$$

Thus the problem is transformed into the standard form in which a sequence of θ^n, $n = 1, 2, \ldots, N$ is to be chosen so as to optimize x_2^N for a process described by equations (2-10) and (2-11). x_1^n is called the primary state variable and x_2^n is the secondary state variable.

Then the Hamiltonian function H^n defined by equation (2-4) can be written as

$$H^n = z_1^n\, T\, (x_1^{n-1};\ \theta^n) + z_2^n \left[x_2^{n-1} + G\, (x_1^{n-1};\ \theta^n)\right] . \tag{2-13}$$

According to equation (2-5), the recurrence relations for the adjoint vector elements z_1 and z_2 are found to be

$$z_1^{n-1} = \frac{\partial H^n}{\partial x_1^{n-1}} = \frac{\partial T\ (x_1^{n-1};\ \theta^n)}{\partial x_1^{n-1}}\ z_1^n + \frac{\partial G\ (x_1^{n-1};\ \theta^n)}{\partial x_1^{n-1}}\ z_2^n , \tag{2-14}$$

$$z_2^{n-1} = \frac{\partial H^n}{\partial x_2^{n-1}} = z_2^n , \tag{2-15}$$

$$n = 1,\ 2,\ \ldots,\ N.$$

Since the objective function is

$$S = \sum_{i=1}^{2} c_i\ x_i^N = x_2^N ,$$

$$c_1 = 0,\ c_2 = 1 .$$

Thus we obtain

$$z_1^N = 0 , \tag{2-15a}$$

$$z_2^N = 1 . \tag{2-15b}$$

Combining equations (2-15b) and (2-15) and substituting in equation (2-14), gives

$$z_2^n = 1, \qquad n = 1, 2, \ldots, N, \tag{2-16}$$

and

$$z_1^{n-1} = \frac{\partial T\ (x_1^{n-1};\ \theta^n)}{\partial x_1^{n-1}}\ z_1^n + \frac{\partial G\ (x_1^{n-1};\ \theta^n)}{\partial x_1^{n-1}}\ , \tag{2-17}$$

$$n = 1, 2, \ldots, N.$$

Combining equations (2-16) and (2-13), we obtain

$$H^n = z_1^n\ T\ (x_1^{n-1};\ \theta^n) + G\ (x_1^{n-1};\ \theta^n) + x_2^{n-1},$$

$$n = 1, 2, \ldots, N.$$

According to equation (2-17), i.e., the stationary condition for optimality, θ^n may be found where

$$\frac{\partial H^n}{\partial \theta^n} = z_1^n \frac{\partial T\ (x_1^{n-1};\ \theta^n)}{\partial \theta^n} + \frac{\partial G\ (x_1^{n-1};\ \theta^n)}{\partial \theta^n} = 0\ .$$

Solving this equation for z_1^n, we obtain

$$z_1^n = - \frac{\dfrac{\partial G\ (x_1^{n-1};\ \theta^n)}{\partial \theta^n}}{\dfrac{\partial T\ (x_1^{n-1};\ \theta^n)}{\partial \theta^n}}\ . \tag{2-18}$$

Substitution of equation (2-18) into equation (2-17) gives the recurrence relation

$$\frac{\dfrac{\partial G\ (x_1^{n-1};\ \theta^n)}{\partial \theta^n}}{\dfrac{\partial T\ (x_1^{n-1};\ \theta^n)}{\partial \theta^n}} = \frac{\dfrac{\partial G\ (x_1^n;\ \theta^{n+1})}{\partial \theta^{n+1}}}{\dfrac{\partial T\ (x_1^n;\ \theta^{n+1})}{\partial \theta^{n+1}}} \cdot \frac{\partial T\ (x_1^n;\ \theta^{n+1})}{\partial x_1^n}$$

$$- \frac{\partial G\ (x_1^n;\ \theta^{n+1})}{\partial x_1^n} ,$$

$$n = 1,\ 2,\ \ldots,\ N-1. \tag{2-19}$$

Combining equations (2-15a) and (2-18) gives

$$\frac{\partial G\ (x_1^{N-1};\ \theta^n)}{\partial \theta^N} = 0\ . \tag{2-20}$$

Making use of the recurrence relation, equation (2-19), along with the performance equations, equation (2-10) and equation (2-20), a number of optimization problems associated with one-dimensional processes can be solved. For processes with a fixed end point x_1^N, condition $z_1^N = 0$ (equation (2-15a)) or equivalently, equation (2-20) is deleted.

3. ORDINARY SAMPLING

By the term ordinary sampling we mean that there is no interaction between sampling and the probability of being defective. In other words sampling is done merely to detect the defective article and it does not have any bearing on the fraction defective. In order to find the best inspection procedure we equate the cost of inspection with the penalty for accepting a defective quantity and the decision is taken accordingly.

Example 1. Optimum Sampling Procedure

The process flow chart of a component may be as shown in Fig. 2.

From the process flow chart we visualize that the component moves from one stage to another (we may consider each process as a stage.) The value of the product changes, and so does the percent defective. The problem is to find the best inspection procedure so that sum of the total expected cost is minimum.

Let

θ^n = Percent sampled at the n^{th} stage, $0 \leqslant \theta^n \leqslant 1$,

$n = 1, 2, \ldots, N$,

a_n = Quantity at the n^{th} stage,

v_n = Penalty for accepting each defective quantity,

f_n = Percent defective at the n^{th} stage,

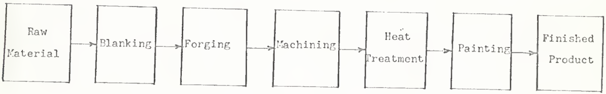

Fig. 2. Process Flow Chart of a Product

I_n = Inspection cost at the n^{th} stage,

x_1^n = Sum of samples up to and including the n^{th} stage,

$$= x_1^{n-1} + a_n\theta^n, \quad n = 1, 2, \ldots, N, \tag{3-1}$$

$x_1^0 = 0$,

x_2^n = Sum of expected cost up to and including the n^{th} stage,

$$= x_2^{n-1} + a_n v_n f_n(1-\theta^n) + a_n I_n \theta^n \tag{3-2}$$

where $a_n v_n f_n(1-\theta^n)$ is the penalty for accepting the defective component and $I_n a_n \theta^n$ is the cost of inspection. It may be pointed out here that these are opposing costs in nature and we want to minimize the sum of these two costs.

The objective is to minimize

$$S = \sum_{i=1}^{2} c_i x_i^N = x_2^N ,$$

where

$$c_1 = 0 \text{ and } c_2 = 1 .$$

Introducing the Hamiltonian function H^n and the adjoint variables z_i^n we may write

$$H^n = z_1^n(x_1^{n-1} + a_n\theta^n) + z_2^n\left[x_2^{n-1} + a_n v_n f_n(1-\theta^n) + a_n I_n\theta^n\right] ,$$

$$n = 1, 2, \ldots, N, \tag{3-3}$$

$$z_1^{n-1} = \frac{\partial H^n}{\partial x_1^{n-1}} = z_1^n , \quad n = 1, 2, \ldots, N, \tag{3-4}$$

$$z_1^N = c_1 = 0 , \tag{3-4a}$$

$$z_2^n = \frac{\partial H^n}{\partial x_2^{n-1}} = z_2^n , \quad n = 1, 2, \ldots, N, \tag{3-5}$$

$$z_2^N = c_2 = 1 . \tag{3-5a}$$

From equations (3-4) and (3-4a) we obtain

$$z_1^n = 0, \quad n = 1, 2, \ldots, N. \tag{3-6}$$

Also from equations (3-5) and (3-5a) we obtain

$$z_2^n = 1, \quad n = 1, 2, \ldots, N. \tag{3-7}$$

Hence the Hamiltonian function can be rewritten as

$$\begin{aligned} H^n &= x_2^{n-1} + a_n v_n f_n (1-\theta^n) + a_n I_n \theta^n \\ &= (x_2^{n-1} + a_n v_n f_n) + (a_n I_n - a_n v_n f_n)\, \theta^n , \end{aligned} \tag{3-8}$$

$$n = 1, 2, \ldots, N.$$

The equation of H^n is linear with respect to x^{n-1} and θ^n; therefore, the strong form of the maximum principle can

be applied, i.e., the objective function is absolutely minimum if and only if H^n is absolutely minimum.

Denoting the variable portion of H^n as H_v^n we may write

$$H_v^n = (a_n I_n - a_n v_n f_n)\ \theta^n$$

where a_n, I_n, v_n and f_n are constants. Hence, the variable portion of the Hamiltonian function H_v^n is a linear function of θ^n.

The optimal value of θ^n which makes H_v^n minimum should occur at the boundary of the admissible region of θ^n, namely, $0 \leqslant \theta^n \leqslant 1$.

The sign of q^n given by

$$q^n = a_n(I_n - v_n f_n)$$

decides which one of the boundaries $\bar{\theta}^n$ lies ($\bar{\theta}^n$ denotes optimum value of θ^n). For a positive value of q^n, $\bar{\theta}^n$ is 0, which is equivalent to no inspection and for a negative value of q^n, $\bar{\theta}^n$ is 1, which is equivalent to inspecting all the components. Summarizing, we have

$$\begin{aligned} &\bar{\theta}^n = 0 \quad &&\text{when } q^n > 0, \\ &\bar{\theta}^n = 1 \quad &&\text{when } q^n < 0, \\ &0 \leqslant \bar{\theta}^n \leqslant 1 \quad &&\text{when } q^n = 0. \end{aligned}$$

In other words

$$\bar{\theta}^n = 0 \quad \text{when } f_n < \frac{I_n}{v_n}\,,$$
$$\bar{\theta}^n = 1 \quad \text{when } f_n > \frac{I_n}{v_n}\,, \tag{3-9}$$
$$0 \leq \bar{\theta}^n \leq 1 \text{ when } f_n = \frac{I_n}{v_n}$$

gives the optimum inspection procedure.

1(a). Numerical example for a 3-stage system.

Let us assume that a component is being produced in 3 stages. The value and cost of inspection at each stage are given as follows:

Stage no.	Value in $	Inspection cost in $
1	5.00	0.05
2	10.00	0.20
3	15.00	1.00

The problem is to decide the inspection procedure to be followed at each stage so that sum of expected cost is minimum.

For first stage.

From equation (3-9) we find that inspection is necessary only when the fraction defective is greater than 0.05/5.00 or 1%. Hence, the decision for the 1st stage is

i) Inspect 100% if the fraction defective is greater than 1%.

ii) Do not inspect if the fraction defective is less than 1%.

iii) May or may not inspect if the fraction defective is equal to 1%.

The decisions for the 2nd and 3rd stages are the same as in the 1st stage, the only changes being in the fractions defective which are 2% and 6.6% respectively.

4. PREVENTATIVE SAMPLING

Preventative sampling differs from ordinary sampling in that the former aims not so much to find the defective quantity as to discourage their future occurence where as the latter concerns primarily in finding defects. It has been found that the probability of being defective is a function of sampling and this, in fact, is the central idea of preventative sampling.

The knowledge that "sampling" is being done encourages a sense of responsibility in the people concerned and makes them more careful. In other words, the more rigorous the sampling the less is the probability of being defective. But the probability of being defective is not a linear function and it contains coefficients like "avoidable" defect, "unavoidable" defect and "elasticity" of reaction. The elasticity of reaction takes into account the human reactions and other intangible reactions that come into play in this type of situation.

Nature of Probability of Defective Function

The relation between $p(\theta)$, i.e., the probability of being defective, and θ is shown in Fig. 3 (9). It will be seen that $p(\theta)$ decreases very rapidly initially with small increase in θ, but thereafter it tends to be constant with larger values of θ. This constant value of $p(\theta)$ which

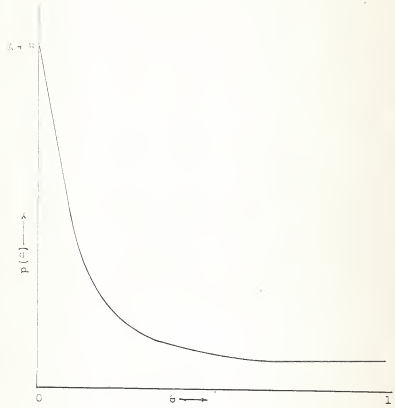

Fig. 3. Relation between θ and $p(\theta)$

cannot be reduced to zero even for very large θ is called the residual "unavoidable defect."

Many attempts have been done to define the probability defective function $p(\theta)$. The equation

$$p(\theta) = A + Be^{-C\theta} \qquad (4\text{-}1)$$

may be used as a fairly good representative of $p(\theta)$ function where A, B and C are constants (9).

We can interpret equation (4-1) as follows: A measures the "unavoidable" defects that cannot be easily avoided even with large value of θ. B measures the "avoidable" defect and it is this defect that we are interested in reducing. C measures the "effect" of sampling on "avoidable" defects. The quantity C can be looked as the "deterrent" effect of sampling on the probability of being defective. It is also regarded as "elasticity" of reaction to sampling. It may be pointed out here that the idea of preventative sampling is based on the existence of the quantity C. The larger the value of C the greater is the scope of improvement by the use of preventative sampling.

5. CASE STUDIES OF OPTIMUM PREVENTATIVE SAMPLING

Example 2. Optimum Preventative Sampling-- Total Sampling Volume Being Given

Let us assume that a manufacturing company is producing N types of products, each being different with respect to the value of the product, probability of being defective, quantity of each type and so on. We have to find the optimum preventative sampling procedure subject to some given constraints. The criterion for the optimum is the one that gives "the total expected value of the undetected faulty articles as small as possible."

Let

a_i = the quantity of the i^{th} type of product,

θ_i = the percentage sampled of the i^{th} type,

v_i = the value of each of the i^{th} product,

$p_i(\theta_i)$ = the probability of being defective.

In general $p(\theta)$ is a monotone decreasing function of θ. Then clearly $a_i(1-\theta_i)$ is the percentage of product not sampled, and $a_i v_i p_i(\theta_i)(1-\theta_i)$ is the value of the undetected defective quantity. The problem is then reduced to the form:

$$\text{Minimize } S = \sum_{i=1}^{N} a_i v_i p_i(\theta_i)(1-\theta_i) \qquad (5\text{-}1)$$

subject to the constraints

$$\sum_{i=1}^{N} a_i\theta_i = D\alpha \qquad (5\text{-}2)$$

and

$$0 \leqslant \theta_i \leqslant 1\ , \quad i = 1,\ 2,\ \ldots,\ N \tag{5-3}$$

where

$$\sum_{i=1}^{N} a_i = D = \text{Total quantity} \tag{5-4}$$

and α is the total percentage sampled, a given fixed quantity. In other words $D\alpha$ gives the overall sampling size.

2(a). Solution by Lagranger's Multiplier

From equations (5-1) and (5-2) and by the use of the Lagrangian multiplier we may write

$$\text{Min } S = \sum_{i=1}^{N} a_i v_i p_i(\theta_i)(1-\theta_i) + \lambda(\Sigma a_i\theta_i - D\alpha). \tag{5-5}$$

$$\text{Let } g_i(\theta_i) = p_i(\theta_i)(1-\theta_i). \tag{5-6}$$

Then equation (5-5) reduces to the form

$$\text{Min } S = \sum_{i=1}^{N} a_i v_i g_i(\theta_i) + \lambda(\sum_{i=1}^{N} a_i\theta_i - D\alpha). \tag{5-7}$$

For minimization we differentiate S with respect to θ_i and equate it to zero

$$\frac{\partial S}{\partial\theta_i} = a_i v_i \frac{\partial g_i(\theta_i)}{\partial\theta_i} + \lambda a_i = 0,$$

or

$$\frac{\partial g_i(\theta_i)}{\partial \theta_i} = -\frac{\lambda}{v_i} . \tag{5-8}$$

The Lagrangian multiplier, λ , is obtained by solving equations (5-2) and (5-8).

Substitution of the value of λ in (5-8) will give us the value of θ_i, the percentage to be sampled at the i^{th} stage, provided $g_i(\theta_i)$ is a known differentiable function of θ_i.

The inherent difficulty of using the Lagrangian multiplier method is present in this problem and it can be noticed that we did not utilize equation (5-3). Hence, only those solutions of equation (5-8) that are non-negative and lie between 0 and 1 are valid. It may be mentioned here that equation (5-8) may give some negative results if α is too small.

Excluding this extreme case, equation (5-8) together with equation (5-2) gives the general relation for the optimum preventative sampling. The relation between $g(\theta)$ and θ is shown in Fig. 4.

From the graph we find $g(\theta)$ decreases first quickly and then slowly. If we analyze equation (5-8) with this point in mind we can conclude that θ_i will be larger as the value of v_i is larger. That is, the higher valued articles will be sampled more intensely.

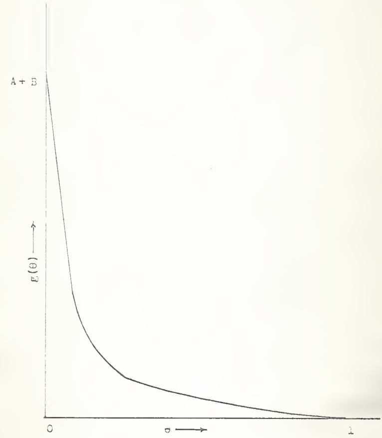

Fig. 4. Relation between θ and g(θ)

Also, for the category in which the probability of defect is high, the derivative of $g(\theta)$ will generally be larger for a given θ. Consequently the corresponding θ satisfying equation (5-8) tends to be larger. This means the categories with high probability of being defective will be sampled more intensely which is desirable.

Now if we assume that the probability of being defective is given by

$$p(\theta) = A + Be^{-C\theta} \tag{5-9}$$

where A, B and C are constants, then from equations (5-6), (5-8) and (5-9) we may write

$$g_i(\theta_i) = \left[A + Be^{-C\theta_i}\right](1-\theta_i) \ , \tag{5-10}$$

$$\frac{\partial g_i(\theta_i)}{\partial \theta_i} = - (A+Be^{-C\theta_i}) - BCe^{-C\theta_i}(1-\theta_i) \ , \tag{5-11}$$

$$-(A+Be^{-C\theta_i}) - BCe^{-C\theta_i}(1-\theta_i) = - \frac{\lambda}{v_i} \ ,$$

$$A+Be^{-C\theta_i} + BCe^{-C\theta_i} - BC\theta_i e^{-C\theta_i} = \frac{\lambda}{v_i} \ ,$$

or

$$A+Be^{-C\theta_i}\left[1+C(1-\theta_i)\right] = \frac{\lambda}{v_i} \ . \tag{5-12}$$

Now if the θ_i are small quantities which is usually true and C is large, then we can approximate equation (5-12) by

$$BCe^{-C\theta_i} = \frac{\lambda}{v_i} \; . \tag{5-13}$$

To obtain equation (5-13) we first approximate equation (5-10) by

$$g_i(\theta_i) = A + Be^{-C\theta_i}$$

and then follow the succeeding steps.

Equation (5-13) together with equation (5-2) will give us the desired solution which is worked out as follows:

Taking the logarithm on both sides of equation (5-13) yields

$$\ln B + \ln C - C\theta_i = \ln\lambda - \ln v_i \; ,$$

or

$$\theta_i = \frac{1}{C}\left[\ln B + \ln C - \ln\lambda + \ln v_i\right] \; . \tag{5-14}$$

Substituting equation (5-14) into equation (5-2), we obtain

$$\sum_{i=1}^{N} a_i \frac{1}{C}\left[\ln B + \ln C - \ln\lambda + \ln v_i\right] = D\alpha \; ,$$

or

$$\frac{\sum_{i=1}^{N} a_i}{DC}\left[\ln \frac{BC}{\lambda}\right] + \frac{\sum_{i=1}^{N} a_i}{DC} \ln v_i = \alpha \; . \tag{5-15}$$

Substituting the value of obtained from equation (5-13) into equation (5-15) and utilizing equation (5-4) we obtain

$$\alpha = \frac{1}{C} + \ln \frac{BC}{v_i BC e^{-C\theta_i}} + \frac{\sum_{i=1}^{N} a_i \ln v_i}{DC}$$

$$= \frac{1}{C}\left[- \ln v_i + C\theta_i \right] + \frac{\sum_{i=1}^{N} a_i \ln v_i}{DC}$$

$$= - \frac{1}{C} \ln v_i + \theta_i + \frac{\sum_{i=1}^{N} a_i \ln v_i}{DC} .$$

or

$$\theta_i = \alpha + \frac{1}{C} \ln v_i - \frac{\sum_{i=1}^{N} a_i \ln v_i}{DC}$$

$$= \alpha + \frac{1}{C} \ln v_i - \frac{1}{C} \sum_{i=1}^{N} a_i/D \ln v_i$$

$$= \alpha + \frac{1}{C} \ln v_i - \frac{1}{C} \sum_{i=1}^{N} \ln(v_i)^{a_i/D}$$

$$= \alpha + \frac{1}{C} \ln \frac{v_i}{\prod_{i=1}^{N} (v_i)^{a_i/D}} .$$

$$= \alpha + \frac{1}{C} \ln \frac{v_i}{v_g} \tag{5-16}$$

where v_g is the geometric mean article value, that is,

$$v_g = \prod_{i=1}^{N} (v_i)^{n_i/D} \tag{5-17}$$

Equation (5-16) gives the optimum preventative sampling procedures provided the different values of θ_i obtained thereby are non-negative and lie between 0 and 1.

It can be seen that the final approximate solution, i.e., equation (5-16) contains only the parameter C.

The approximation is equivalent to neglecting the term $(1-\theta_i)$ in the equation $g(\theta) = (1-\theta)\ p(\theta)$, i.e., we are neglecting the improvement attained by direct detection of defects for that attained by preventative means.

2(b). Solution by the Discrete Maximum Principle

We may consider each of the different types of product as a stage having different values, different probability of being defective, etc.

Let us define

θ^n = Percent of articles sampled at the n^{th} stage, $0 \leqslant \theta^n \leqslant 1$, $n = 1, 2, \ldots, N$,

a_n = Quantity at the n^{th} stage,

$p_n(\theta^n)$ = Probability of being defective at the n^{th} stage,

v_n = Value of each article at the n^{th} stage,

x_1^n = Sum of samples up to and including the n^{th} stage,

x_2^n = Sum of expected value of undetected defective quantity up to and including n^{th} stage.

Then the performance equations are

$$x_1^n = x_1^{n-1} + a_n \theta^n, \quad n = 1, 2, \ldots, N, \tag{5-18}$$

$$x_1^0 = 0, \tag{5-19}$$

$$x_1^N \leqslant D\alpha , \tag{5-20}$$

$$x_2^n = x_2^{n-1} + a_n v_n p_n(\theta^n)(1-\theta^n), \; n = 1, 2, \ldots, N. \tag{5-21}$$

The objective is to minimize

$$S = \sum_{i=1}^{2} c_i x_i^N = x_2^N , \tag{5-22}$$

where

$$c_1 = 0 \text{ and } c_2 = 1 . \tag{5-23}$$

Introducing the Hamiltonian function H^n and adjoint variable z_i^n we may write

$$H^n = z_1^n(x_1^{n-1}+a_n\theta^n) + z_2^n\left[(x_2^{n-1}+a_n v_n p_n(\theta^n)(1-\theta^n)\right], \tag{5-24}$$

$$n = 1, 2, \ldots, N,$$

$$z_1^{n-1} = \frac{\partial H^n}{\partial x_1^{n-1}} = z_1^n , \quad n = 1, 2, \ldots, N, \tag{5-25}$$

$$z_1^N \neq c_1 ,$$

$$z_2^{n-1} = \frac{\partial H^n}{\partial x_2^{n-1}} = z_2^n , \quad n = 1, 2, \ldots, N, \tag{5-26}$$

$$z_2^N = c_2 = 1 . \tag{5-27}$$

From equations (5-26) and (5-27) we find

$$z_2^n = 1 , \quad n = 1, 2, \ldots, N . \tag{5-28}$$

Hence equation (5-24) reduces to

$$H^n = z_1^n(x_1^{n-1}+a_n\theta^n) + x^{n-1} + a_n v^n p_n(\theta^n)(1-\theta^n). \tag{5-29}$$

The variable portion of equation (5-29) denoted by H_v^n is

$$H_v^n = z_1^n \, a_n\theta^n + a_n v_n p_n(\theta^n)(1-\theta^n) . \tag{5-30}$$

Now, as in the Lagrangian multiplier case, let us assume

$$g_n(\theta^n) = p_n(\theta^n)(1-\theta^n) .$$

Then equation (5-30) is transformed into

$$H_v^n = z_1^n \, a_n\theta^n + a_n v_n g_n(\theta^n) . \tag{5-31}$$

The optimum θ^n is obtained by differentiating H_v^n partially with respect to θ^n and equating it to zero.

$$\frac{\partial H_v^n}{\partial \theta^n} = z_1^n a_n + a_n v_n \frac{\partial g_n(\theta^n)}{\partial \theta^n} = 0 ,$$

or

$$\frac{\partial g_n(\theta^n)}{\partial \theta^n} = - \frac{z_1^n}{v_n} . \tag{5-32}$$

Equation (5-32) is the same as equation (5-8), found earlier by employing the Lagrangian multiplier method. The difference is that λ , the Lagrangian multiplier, has been replaced by z_1^n, the adjoint variable of the discrete maximum principle.

Now, as done earlier, let us assume

$$p_n(\theta^n) = A + Be^{-C\theta^n}, \quad n = 1, 2, \ldots, N. \tag{5-33}$$

Then equation (5-22) is written as

$$x_2^n = x_2^{n-1}+a_n v_n(A+Be^{-C\theta^n})(1-\theta^n), \; n = 1, 2, \ldots, N, \tag{5-34}$$

$$x_2^0 = 0 .$$

Comparing equations (5-19) and (5-34) with performance equations of the one-dimensional process we find

$$T^n \; (x_1^{n-1}; \; \theta^n) = x_1^{n-1} + a_n\theta^n ,$$

$$G^n\ (x_1^{n-1};\ \theta^n) = a_n v_n (A+Be^{-C\theta^n})(1-\theta^n)\ .$$

Making approximations, i.e., neglecting the term $(1-\theta_i)$ from the above equation, which in turn is equivalent to neglecting the improvement attained by direct detection of defect for that attained by preventative means, we may write

$$\frac{\partial G(x_1^{n-1};\ \theta^n)}{\partial \theta^n} = -a_n v_n BCe^{-C\theta^n}\ , \qquad (5\text{-}35)$$

$$\frac{\partial G(x_1^{n-1};\ \theta^n)}{\partial x_1^{n-1}} = 0\ , \qquad (5\text{-}36)$$

$$\frac{\partial T(x_1^{n-1};\ \theta^n)}{\partial \theta^n} = a_n\ , \qquad (5\text{-}37)$$

$$\frac{\partial T(x_1^{n-1};\ \theta^n)}{\partial x_1^{n-1}} = 1\ . \qquad (5\text{-}38)$$

Substituting these partial derivatives in the recurrence relation of the one dimensional process given by equation (2-19), we obtain

$$\frac{-a_n v_n BCe^{-C\theta^n}}{a_n} = \frac{-a_{n+1} v_{n+1} BCe^{-C\theta^{n+1}}}{a_{n+1}}\ (1) - 0\ ,$$

$$n = 1, 2, \ldots, N-1 ,$$

or

$$v_n e^{-C\theta^n} = v_{n+1} e^{-C\theta^{n+1}} ,$$

or

$$\theta^n = \theta^{n+1} - \frac{1}{C} \ln \frac{v_{n+1}}{v_n} , \quad n = 1, 2, \ldots, N-1 . \qquad (5\text{-}39)$$

This is the recurrence relation of the optimal decision.

With the help of equations (5-18), (5-19), (5-20) and (5-39) we may obtain the value of θ^n in terms of known quantities. This is worked out as follows:

Let E be an assumed value of x_1^1. For n = 1 equation (5-18) becomes

$$x_1^1 = x_1^0 + a_1 \theta^1$$

Since $x_1^0 = 0$ and $x_1^1 = E$, we obtain

$$E = a_1 \theta^1 ,$$

or

$$\theta^1 = \frac{E}{a_1} \qquad (5\text{-}40)$$

Substituting equation (5-40) into equation (5-39) for n = 1 yields

$$\theta^1 = \theta^2 - \frac{1}{C} \ln \frac{v_2}{v_1} ,$$

or

$$\theta^2 = \frac{E}{a_1} + \frac{1}{C} \ln \frac{v_2}{v_1} \qquad (5\text{-}41)$$

For n = 2, equation (5-18) becomes

$$x_1^2 = x_1^1 + a_2 \theta^2$$

$$= E + a_2 \left(\frac{E}{a_1} + \frac{1}{C} \ln \frac{v_2}{v_1}\right) ,$$

$$= E \left(1 + \frac{a_2}{a_1}\right) + \frac{1}{C} a_2 \ln \frac{v_2}{v_1} . \qquad (5\text{-}42)$$

Again from equation (5-39) for n=2, we obtain

$$\theta^2 = \theta^3 - \frac{1}{C} \ln \frac{v_3}{v_2} ,$$

or

$$\theta^3 = \frac{E}{a_1} + \frac{1}{C} \left(\ln \frac{v_2}{v_1} + \ln \frac{v_3}{v_2}\right)$$

$$= \frac{E}{a_1} + \frac{1}{C} \ln \frac{v_3}{v_1} . \qquad (5\text{-}43)$$

Also from equation (5-18) for n=3, we obtain

$$x_1^3 = x_1^2 + a_3\theta^3$$

$$= E\ (1 + \frac{a_2}{a_1}) + a_2 \ln \frac{v_2}{v_1} + a_3\ (\frac{E}{a_1} + \frac{1}{C} \ln \frac{v_3}{v_1})$$

$$= E\ (1 + \frac{a_2}{a_1} + \frac{a_3}{a_1}) + \frac{1}{C}\ (a_2 \ln \frac{v_2}{v_1} + a_3 \ln \frac{v_3}{v_1})\ .$$

Similarly we obtain

$$x_1^N = E\ (1 + \frac{a_2}{a_1} + \frac{a_3}{a_1} + \ldots, + \frac{a_N}{a_1}) + \frac{1}{C}\ (a_2 \ln \frac{v_2}{v_1} +$$

$$a_3 \ln \frac{v_3}{v_1} + a_4 \ln \frac{v_4}{v_1} + \ldots, + a_N \ln \frac{v_N}{v_1}$$

$$= E\ \frac{1}{a_1}(\sum_{n=1}^{N} a_n) + \frac{1}{C}\left[\sum_{n=1}^{N}(a_n \ln v_n) - \ln v_1\ (\sum_{n=1}^{N} a_n)\right] \tag{5-44}$$

Combining equations (5-20) and (5-44), yields

$$E\ \frac{1}{a_1}(\sum_{n=1}^{N} a_n) + \frac{1}{C}\left[\sum_{n=1}^{N}(a_n \ln v_n) - \ln v_1\ (\sum_{n=1}^{N} a_n)\right] = D\alpha,$$

or

$$E = \frac{D\alpha - \frac{1}{C}\left[\sum_{n=1}^{N}(a_n \ln v_n) - \ln v_1 \left(\sum_{n=1}^{N} a_n\right)\right]}{\frac{1}{a_1}\left(\sum_{n=1}^{N} a_n\right)} . \qquad (5\text{-}45)$$

Now from equations (5-40) and (5-45) we write

$$a_1\theta^1 = \frac{D\alpha - \frac{1}{C}\left[\sum_{n=1}^{N}(a_n \ln v_n) - \ln v_1 \left(\sum_{n=1}^{N} a_n\right)\right]}{\frac{1}{a_1}\left(\sum_{n=1}^{N} a_n\right)} ,$$

or

$$\theta^1 = \frac{D\alpha}{\sum_{n=1}^{N} a_n} + \frac{1}{C}\left[\ln v_1 - \frac{\sum_{n=1}^{N}(a_n \ln v_n)}{\sum_{n=1}^{N} a_n}\right]$$

or generalizing

$$\theta^n = \alpha + \frac{1}{C}\left[\ln v_n - \sum_{n=1}^{N} \ln (v_n)^{a_n/D}\right]$$

$$= \alpha + \frac{1}{C} \ln \frac{v_n}{\prod_{n=1}^{N} (v_n)^{a_n/D}} .$$

$$= \alpha + \frac{1}{C} \ln \frac{v_n}{v_g} \qquad n = 1, 2, \ldots, N \qquad (5\text{-}46)$$

where $v_g = \prod_{n=1}^{N} (v_n)^{a_n/D}$ is the geometric mean article value.

2(c). Numerical Example

A 5 percent sample of a group of 1000 articles of five different types is taken. The price of each type of article and their quantities are as follows:

Type n	Value v_n	Quantity a_n
1	5	400
2	10	250
3	15	100
4	20	150
5	25	100

Also it is known that an increase of 10% of the sampling fraction reduces the avoidable defect by $\frac{1}{12}$. Find the optimum preventative sampling procedure.

From the problem we know

$$e^{-0.1C} = \frac{1}{12}$$

or $0.1 \times C \log_{10} e = \log 1 - \log_{10} 12$

$$\text{or} \quad C = \frac{\log_{10} 12}{0.1 \log_{10} e} = 24.84$$

The geometric mean article value

$$v_g = \prod_{i=1}^{N} (v_i^{\frac{n_i}{N}})$$

$$= (5)^{\frac{400}{1000}} (10)^{\frac{250}{1000}} (15)^{\frac{100}{1000}} (20)^{\frac{150}{1000}} (25)^{\frac{100}{1000}}$$

$$= (5)^{0.4} (10)^{.25} (15)^{.1} (20)^{.15} (25)^{0.1}$$

$$= 9.596$$

Employing equation (5-46) where $\alpha = .05$ (given) we can write

$$\theta_1 = \alpha + \frac{1}{C} \log_e \frac{v_1}{v_g}$$

$$= .05 + \frac{1}{24.84} \ln \frac{5}{9.596}$$

$$= .05 - .0262 = .0238$$

$$\theta_2 = \alpha + \frac{1}{C} \ln \frac{v_2}{v_g}$$

$$= .05 + \frac{1}{24.84} \ln \frac{10}{9.596}$$

$$= .05 + .0016 = .0516$$

$$\theta_3 = \alpha + \frac{1}{C} \ln \frac{v_3}{v_g}$$

$$= .05 + \frac{1}{24.84} \ln \frac{15}{9.596}$$

$$= .05 + .0179 = .0679$$

$$\theta_4 = \alpha + \frac{1}{C} \ln \frac{v_4}{v_g}$$

$$= .05 + \frac{1}{24.84} \ln \frac{20}{9.596}$$

$$= .05 + .0295 = .0795$$

$$\theta_5 = \alpha + \frac{1}{C} \ln \frac{v_5}{v_g}$$

$$= .05 + \frac{1}{24.84} \ln \frac{25}{9.596}$$

$$= .05 + .0385 = .0885$$

Rounding off to 3 decimal places we get the answers as

$\theta_1 = 2.4\%$

$\theta_2 = 5.2\%$

$\theta_3 = 6.8\%$

$\theta_4 = 8.0\%$

$\theta_5 = 8.9\%$

It may be verified that

$$\sum_{n=1}^{5} a_n\theta^n = (400)(.024)+(250)(.052)+(100)(.068)+(150)(.08)$$

$$+(100)(.089)$$

$$= 50 \text{ , i.e., } 5\% \text{ of } 1000.$$

Example 3. Optimum Preventative Sampling
Considering the Cost of Inspection

Let there be a total of D articles of N different types which have been categorized into N stages so that each category has the same value of the article, the same probability of being defective, and the same cost of inspection. We have to optimize the sampling procedure of each stage so that the expected total cost is minimum.

Let

θ^n = Percent sampled at the n^{th} stage,

a_n = Quantity at the n^{th} stage,

v_n = Value of each article at the n^{th} stage,

$p_n(\theta^n)$ = Probability of being defective at the n^{th} stage,

I_n = Cost of inspection of each article at the n^{th} stage,

x_1^n = Sum of samples up to and including the n^{th} stage

$$= x_1^{n-1} + a_n\theta^n \text{ ,} \qquad (5\text{-}47)$$

x_2^n = Sum of expected cost up to and including the n^{th} stage

$$= x^{n-1} + a_n v_n p_n(\theta^n)(1-\theta^n) + I_n a_n \theta^n \qquad (5\text{-}48)$$

where $a_n v_n p_n(\theta^n)(1-\theta^n)$ is the cost of undetected defects and $I_n a_n \theta^n$ is the cost of inspection.

The objective is to minimize

$$S = \sum_{i=1}^{2} c_1 x_i^N = x_2^N \qquad (5\text{-}49)$$

where

$$c_1 = 0 \quad c_2 = 1.$$

Introducing the Hamiltonian function H^n and the adjoint variables z_i^n we may write

$$H^n = z_1^n(x_1^{n-1}+a_n\theta^n) + z_2^n\,(x_2^{n-1}+a_n v_n p_n \theta^n(1-\theta^n)+I_n a_n \theta^n) \quad , \qquad (5\text{-}50)$$

$$n = 1,\ 2,\ \ldots,\ N\ ,$$

$$z_1^{n-1} = \frac{\partial H^n}{\partial x_1^{n-1}} = z_1^n\ , \quad n = 1,\ 2,\ \ldots,\ N \qquad (5\text{-}51)$$

$$z_1^N = c_1 = 0\ , \qquad (5\text{-}51a)$$

$$z_2^{n-1} = \frac{\partial H^n}{\partial x_2^{n-1}} = z_2^n , \quad n = 1, 2, \ldots, N, \tag{5-52}$$

$$z_2^N = c_2 = 1 . \tag{5-52a}$$

From equations (5-51) and (5-51a) we get

$$z_1^n = 0, \quad n = 1, 2, \ldots, N . \tag{5-53}$$

Also from equations (5-52) and (5-52a) we obtain

$$z_2^n = 1, \quad n = 1, 2, \ldots, N . \tag{5-54}$$

Substituting the values of z_1^n and z_2^n in equation (5-50) we obtain

$$H^n = x_2^{n-1} + a_n v_n p_n(\theta^n)(1-\theta^n) + I_n a_n \theta^n . \tag{5-55}$$

Therefore, the variable portion of the Hamilton function denoted by H_v^n is

$$H_v^n = v_n a_n p_n(\theta^n)(1-\theta^n) + I_n a_n \theta^n \tag{5-56}$$

$$= v_n a_n g_n(\theta^n) + I_n a_n \theta^n$$

where $g_n(\theta^n) = p_n(\theta^n)(1-\theta^n)$.

The optimum θ^n may be obtained by differentiating H_v^n partially with respect to θ^n and equating it to zero.

$$\frac{\partial H_v^n}{\partial \theta^n} = v_n a_n \frac{\partial g_n(\theta^n)}{\partial \theta^n} + I_n a_n = 0 \tag{5-57}$$

or

$$-\frac{\partial g_n(\theta^n)}{\partial \theta^n} = \frac{I_n}{v_n} . \tag{5-58}$$

Equation (5-58) gives the general relation for the optimum preventative sampling considering inspection cost.

3(a) Exact solution

Let the probability of being defective be

$$p_n(\theta^n) = A + Be^{-C\theta^n}$$

where A, B, and C are certain constants which have been defined earlier. Then we have

$$g_n(\theta^n) = p_n(\theta^n)(1-\theta^n) = (A+Be^{-C\theta^n})(1-\theta^n) ,$$

$$\frac{\partial g_n(\theta^n)}{\partial \theta^n} = -(A+Be^{-C\theta^n}) - BCe^{-C\theta^n}(1-\theta^n) .$$

Substituting this equation into equation (5-58) we obtain

$$A + Be^{-C\theta^n} + BCe^{-C\theta^n}(1-\theta^n) = \frac{I_n}{v_n} ,$$

$$A + Be^{-C\theta^n}\left[1 + C(1-\theta^n)\right] = \frac{I_n}{v_n} ,$$

or

$$e^{-C\theta^n}(1 + C - C\theta^n) = \frac{\frac{I_n}{v_n} - A}{B}, \quad n = 1, 2, \ldots, N. \quad (5\text{-}59)$$

The solution of this equation gives the preventative sampling procedure considering the inspection cost.

3(b) Approximate solution

The approximation is the same as earlier which is equivalent to neglecting the term $(1-\theta^n)$ in

$$g_n(\theta^n) = p_n(\theta^n)(1-\theta^n)$$

which is a fairly good approximation as θ^n is usually very small. Then we may write

$$g_n(\theta^n) = A + Be^{-C\theta^n}$$

$$\frac{\partial g_n(\theta^n)}{\partial \theta^n} = - BCe^{-C\theta^n} .$$

Substituting in equation (5-58) yields

$$BCe^{-C\theta^n} = \frac{I_n}{v_n} . \quad (5\text{-}60)$$

Now taking the logarithm on both sides of equation (5-60), we obtain

$$\ln B + \ln C - C\theta^n = \ln I_n - \ln v_n ,$$

or

$$C\theta^n = \ln \frac{BC}{I_n/v_n} ,$$

or

$$\theta^n = \frac{1}{C} \ln \frac{BC}{I_n/v_n} , \quad n = 1, 2, \ldots, N. \tag{5-61}$$

Equation (5-61) is an approximate solution of optimum preventative sampling considering inspection cost.

3(c). Numerical Example

Suppose a manufacturing company produces 3 types of products. The cost of inspection and value of each product is given in the following table.

Type of Product	Value in $	Inspection cost in $
1	5.00	0.05
2	10.00	0.20
3	15.00	0.75
4	20.00	2.00
5	25.00	3.75
6	30.00	7.50
7	35.00	17.50

We have to find the optimum preventative sampling procedure. Given that probability of being defective is

$$p(\theta) = A + Be^{-C\theta}$$

where A, B, C are certain constnants. As discussed earlier, A measures the "unavoidable defect," B measures the "avoidable defect" and C measures the "effect" of sampling on avoidable defect. The higher the value of C, the greater is the scope of improvement by preventative sampling. The problem is worked out by exact solution (Equation (5-59)) and by approximate solution (Equation (5-61)) for different values of C but for some fixed inspection cost. For the product type 1, the value is \$5.00 whereas, the inspection cost is \$0.05. Hence we may consider it as 1% inspection cost. In other words we are defining the cost of inspection as a percentage of the value of the article. A solution is given for 7 different inspection costs namely 1%, 2%, 5%, 10%, 15%, 25%, and 50%.

Figure 5 shows the effect of C on the exact and approximate solution for constant inspection cost. Figure 6 shows the effect of C on the percentage difference between the exact and approximate solution.

Analyzing the results we observe the following points:

i) The higher the inspection cost the lower the fraction to be sampled for the same value of C (Fig. 6). This is reasonable as for a lower inspection cost we can afford to take a higher fraction to be sampled and balance it with the cost of accepting defective material.

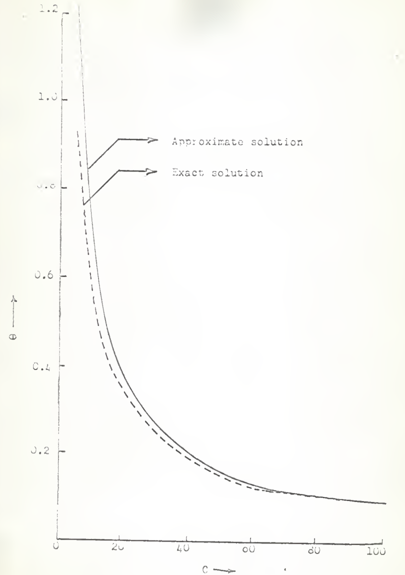

Fig. 5. θ versus C for Approximate and Exact Solution at 1% Inspection Cost

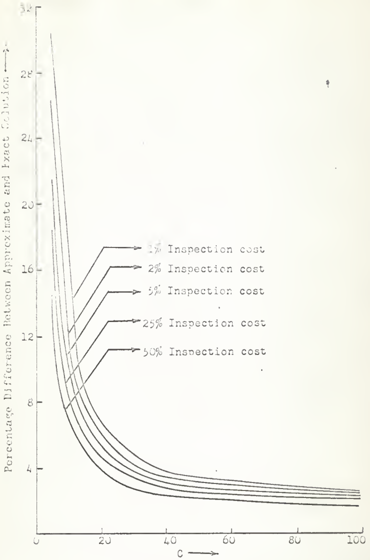

Fig. 6. Relation between C and percentage difference between exact and approximate solution

ii) The greater the value of C, the deterent effect on avoidable defect, the lower is the fraction to be sampled for the same inspection cost and vice versa (Fig. 5).

iii) At higher values of C, the approximate solution is very close to the exact solution. [illegible] values of C greater than 25, the percentage difference between the approximate solution and the exact solution is no more than 5% (Fig. 6). At low values of C, say 5, the approximate solution deviates considerable from the exact solution.

iv) For low values of C and low inspection cost the sampling fraction is almost 100% for the exact solution and it may be more than 100% in the case of an approximate solution which is an infeasible solution. This fact points out that we cannot use an approximate solution for low values of C. Table 1 presents the fraction to be sampled obtained by approximate and exact solution for different values of C and different inspection cost. The percentage difference between approximate and exact solutions is also shown in the Table.

Summarizing we may say that an approximate solution may be used for C 25.

Table 1. Optimum Preventative Sampling Solutions by Exact and Approximate Method for Different Values of C and Different Inspection Cost

C	Inspection Cost	Approximate Solution	Exact Solution	Percentage Difference
	1%	1.2241	0.9370	30.60
	2%	1.0854	0.8590	26.40
	5%	0.9022	0.7420	21.60
5	10%	0.7635	0.6440	18.55
	15%	0.6824	0.5840	16.85
	25%	0.5803	0.5060	14.70
	50%	0.4417	0.3970	11.30
	1%	0.6813	0.6020	13.18
	2%	0.6120	0.5470	11.90
	5%	0.5204	0.4710	10.50
10	10%	0.4511	0.4120	9.50
	15%	0.4105	0.3770	8.90
	25%	0.3595	0.3320	8.30
	50%	0.2901	0.2710	7.05
	1%	0.4813	0.4430	8.65
	2%	0.4351	0.4040	7.70
	5%	0.3740	0.3500	6.85
15	10%	0.3278	0.3080	6.42
	15%	0.3007	0.2830	6.25
	25%	0.2667	0.2520	5.84
	50%	0.2205	0.2100	5.00

Table 1. Continued

C	Inspection Cost	Approximate Solution	Exact Solution	Percentage Difference
	1%	0.3753	0.3520	6.62
	2%	0.3407	0.3220	5.80
	5%	0.2949	0.2800	5.32
20	10%	0.2602	0.2480	4.90
	15%	0.2399	0.2290	4.75
	25%	0.2144	0.2050	4.60
	50%	0.1797	0.1730	3.87
	1%	0.3092	0.2930	5.52
	2%	0.2815	0.2690	4.65
	5%	0.2448	0.2350	4.16
25	10%	0.2171	0.2090	3.88
	15%	0.2009	0.1930	4.09
	25%	0.1804	0.1740	3.68
	50%	0.1527	0.1480	3.18
	1%	0.2637	0.2520	4.65
	2%	0.2406	0.2310	4.15
	5%	0.2101	0.2030	3.50
30	10%	0.1870	0.1810	3.32
	15%	0.1735	0.1680	3.27
	25%	0.1564	0.1520	2.90
	50%	0.1333	0.1290	3.33

Table 1. Continued

C	Inspection Cost	Approximate Solution	Exact Solution	Percentage Difference
	1%	0.1685	0.1630	3.37
	2%	0.1546	0.1500	3.07
	5%	0.1363	0.1300	2.48
50	10%	0.1224	0.1200	2.00
	15%	0.1143	0.1120	2.05
	25%	0.1041	0.1020	2.05
	50%	0.0902	0.0880	2.50
	1%	0.0920	0.0900	2.22
	2%	0.0850	0.0830	2.40
	5%	0.0757	0.0740	2.30
99	10%	0.0687	0.0680	1.03
	15%	0.0646	0.0630	2.54
	25%	0.0595	0.0580	2.59
	50%	0.0525	0.0520	0.96

Example 4. Optimum Preventative Sampling--the Cost of Inspection and the Total Sampling Volume being given

The system considered is the same as in Example 3 but the total sampling volume is a given fixed quantity. This type of situation is generally encountered in practice as the sampling capacity is limited due to men and machine. We have to optimize the sampling procedure of each stage so that the expected total cost is minimum subject to the given constraint of total sampling volume.

Let us define

θ^n = Percent of articles sampled at the n^{th} stage,

$0 \leqslant \theta^n \leqslant 1, \quad n = 1, 2, \ldots, N,$

a_n = Quantity at the n^{th} stage,

$p_n(\theta^n)$ = Probability of being defective at the n^{th} stage,

v_n = Value of each article at the n^{th} stage,

x_1^n = Sum of samples up to and including the n^{th} stage,

x_2^n = Sum of expected value of undetected defective quantity up to and including the n^{th} stage,

Then the performance equations are

$$x_1^n = x_1^{n-1} + a_n\theta^n, \qquad n = 1, 2, \ldots, N, \tag{5-62}$$

$$x_1^0 = 0, \tag{5-63}$$

$$x_1^N \leqslant D\alpha , \tag{5-64}$$

$$x_2^n = x_2^{n-1} + a_n\, v_n\, p_n(\theta^n)\,(1-\theta^n) + I_n\, a_n\, \theta^n$$

$$= x_2^{n-1} + a_n\, v_n\,(A+Be^{-C\theta^n})(1-\theta^n) + I_n\, a_n\, \theta^n \qquad (5\text{-}65)$$

$$n = 1, 2, \ldots, N.$$

$$x_2^0 = 0$$

Since

$$p_n(\theta^n) = A+Be^{-C\theta^n}$$

also

$$\sum_{n=1}^{N} a_n = D = \text{Total quantity}$$

and α is total percentage sampled, a given fixed quantity. In other words $D\alpha$ gives the overall sampling size. Comparing equations (5-62) and (5-65) with performance equations of the one-dimensional process we find

$$T^n(x_1^{n-1};\, \theta^n) = x_1^{n-1} + a_n\theta^n,$$

$$G^n(x_1^{n-1};\, \theta^n) = a_n v_n (A+Be^{-C\theta^n})(1-\theta^n) + I_n a_n \theta^n\ .$$

Making approximations, i.e., neglecting the term $(1-\theta^n)$ from the above equation which in turn is equivalent to

neglecting the improvement attained by direct detection of defect for that attained by preventative means we may write

$$\frac{\partial G\ (x_1^{n-1};\ \theta^n)}{\partial \theta^n} = -\ a_n v_n BCe^{-C\theta^n} + I_n a_n, \qquad (5\text{-}66)$$

$$\frac{\partial G\ (x_1^{n-1};\ \theta^n)}{\partial x_1^{n-1}} = 0\ , \qquad (5\text{-}67)$$

$$\frac{\partial T\ (x_1^{n-1};\ \theta^n)}{\partial \theta^n} = a_n\ , \qquad (5\text{-}68)$$

$$\frac{\partial T\ (x_1^{n-1};\ \theta^n)}{\partial x_1^{n-1}} = 1\ . \qquad (5\text{-}69)$$

Substituting these partial derivatives in the recurrence relation of the one-dimensional process given by equation (2-19) we obtain

$$\frac{-a_n v_n BCe^{-C\theta^n} + I_n a_n}{a_n} = \frac{-a_{n+1} v_{n+1} BCe^{-C\theta^{n+1}} + I_{n+1} a_{n+1}(1)}{a_{n+1}} - 0,$$

or

$$BC\ v_n e^{-C\theta^n} + I_n = BCv_{n+1} e^{-C\theta^{n+1}} + I_{n+1}$$

$$n = 1,\ 2,\ \ldots,\ N-1$$

or

$$BCv_{n+1}e^{-C\theta^{n+1}} = BCe^{-C\theta^{n}} + I_n - I_{n+1} \qquad (5\text{-}70)$$

The decision at each stage is dependent on the decision taken at the preceeding stage. Noting that all terms of the right hand side are known if θ^n is known, and we need to find θ^{n+1}, let

$$E^n = BCe^{-C\theta^n} + I_n = I_{n+1} \text{ .}$$

We obtain

$$e^{-C\theta^{n+1}} = \frac{E^n}{BCv_{n+1}}$$

$$-C\theta^{n+1} = \ln\left(\frac{E^n}{BCv_{n+1}}\right) \text{ ,}$$

or

$$\theta^{n+1} = \frac{1}{C}\ln\left(\frac{BCv_{n+1}}{E^n}\right) \qquad n = 1, 2, \ldots, N\text{-}1 \qquad (5\text{-}71)$$

Equation (5-71) is the recurrence relation of the optimum decision variable. This recurrence relation can be used to obtain the sequence of θ^n which will minimize the total expected cost for any given total sampling volume.

4(a) Computational Procedure

Step 1. Assume $\theta^1 = 0$.

Step 2. Compute $\theta^2, \theta^3, \ldots, \theta^N$ from equation (5-71).

Step 3. Compute $x_1^N = \sum_{n=1}^{N} a_n \theta^n - D\alpha$.

x_1^N will be in one of the following situations: (a) less than zero, (b) equal to zero, (c) greater than zero. If it is (a) then go to Step 4, if it is (b) then we have reached the optimal stage, go to Step 6, if it is (c) then go to Step 5.

Step 4. Increment θ^1 by 0.01 and go to Step 2.

Step 5. Decrease θ^1 by 0.0002 and go to Step 2 until x_1^N is again less than zero; when x_1^N is less than zero then go to Step 6.

Step 6. The solution has reached the optimal stage, and the values of θ^n for $n = 1, 2, \ldots, N$ are the optimum decisions for each stage.

4(b) Numerical Example

A 10% sample of a group of 1,000 articles of five different types is taken. The price of each type of article, cost of inspection and their quantities are as follows:

Type n	Value v_n	Inspection Cost I_n	Quantity a_n
1	5	0.05	400
2	10	0.20	250
3	15	0.45	100
4	20	0.80	150
5	25	1.25	100

Using the recurrence relation (5-71) and the end condition (5-64) the optimum sampling procedure obtained is as follows:

$\theta^1 = 0.0732$

$\theta^2 = 0.1013$

$\theta^3 = 0.1180$

$\theta^4 = 0.1303$

$\theta^5 = 0.1403$

Rounding off to 3 decimal places we get the answers as

$\theta^1 = 7.3\%$

$\theta^2 = 10.1\%$

$\theta^3 = 11.8\%$

$\theta^4 = 13.0\%$

$\theta^5 = 14.0\%$

It may be verified that

$$\sum_{n=1}^{5} a_n\theta^n = (400)(.73) + (250)(.101) + (100)(.118) + (150)(.13) + (100)(.14)$$

$$= 99.8 \approx 100, \text{ i.e., } 10\% \text{ of } 1{,}000.$$

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REFERENCES

1. Chaudhari, D. R., "Equipment Replacement by the Continuous Maximum Principle," M.S. report, Kansas State University, Manhattan, Kansas, 1967.

2. Chung, H.C., "Optimum Design of a Gear-train by the Discrete Maximum Principle," M.S. report, Kansas State University, Manhattan, Kansas, 1967.

3. Fan, L. T., and Wang, C. S., "The Discrete Maximum Principle--A Study of Multistage Systems Organizations," New York, John Wiley and Sons, 1964.

4. Fan, L. T., Wang, C. S., Tillman, F. A., and Hwang, C. L., "Optimization of System Reliability," IEEE Transactions of Reliability (in press, 1967).

5. Hwang, C. L., Fan, L. T., and Erickson, L. E., "Optimum Production Planning by the Maximum Principle," Management Science, 13, 751-755 (1967).

6. Hwang, C. L., Chen, S. K., Panchal, J. M., Schrader, G. F., and Fan, L. T., "The Application of the Discrete Maximum Principle to Transportation Problems with Linear and Non-Linear Cost Functions." Special Report No. 50, Kansas Engineering Experiment Station, Kansas State University Bulletin, (1966).

7. Juran, J. M., "Quality Control Handbook," McGraw Hill, 1951.

8. Seder, L. A., "The Technique of Preventing Defects," J of Industrial Quality Control, 6, 23-30 (1953).

9. Whittle, P., "Optimum Preventative Sampling," Operations Research, 2, 197-204 (1954).

10. Yang, T. C., and Snell, R. R., "Traffic Assignment by the Maximum Principle," Proceedings of the American Society of Civil Engineers, 1966.

OPTIMUM PREVENTATIVE SAMPLING BY THE DISCRETE MAXIMUM PRINCIPLE

by

SUBIR BANERJI

B. S. (Mathematics) University of Calcutta
Calcutta, India, 1959

B. S. (Mechanical Engineering) Government College of
Engineering & Technology, Raipur, India, 1964

AN ABSTRACT OF A MASTER'S REPORT

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The objective of this report is to investigate optimum preventative quality control procedures and to show the applicability of the discrete version of the maximum principle in this type of problem.

It is recognized that the idea of sampling is not only to find defects but also to prevent their future occurence. This is in recognition of the principle that it is better to prevent defects from occuring than to let them occur and then to make the best of it.

Preventative sampling aims not so much to find the defective quantity as to discourage its future occurence. The knowledge that "sampling" is being done instills a sense of responsibility in the people concerned and deters them from making mistakes. The more rigorous the sampling the less is the probability of being defective. But the probability of being defective is not a linear function of sampling and it contains coefficients like "avoidable" defect, "unavoidable" defect and "elasticity" of reaction. The elasticity of reaction takes into account the human reactions and other intangible reactions that come into play in this type of situation.

An "ordinary sampling" problem is solved where the objective is to minimize the total expected cost. The sampling decision is found to be dependent on the ratio of inspection cost to the value of the article.

Three different types of preventative sampling problems are then solved, the objective in each case being to minimize the total expected cost. The sampling procedure to be followed, as given by an approximate solution for each stage in a situation with a given sampling volume, is dependent on the logarithm of the ratio of the value of the article in that stage to the geometric mean article value.

Both exact and approximate solutions are developed for cases with given inspection cost, and it is found that an approximate solution can advantageously be employed in cases where the value of C, the elasticity of reaction, is greater than 25 and that an approximate solution cannot be employed for low values of C, say 5, as it deviates considerably from the exact solution.

A general type of solution for N-stages is solved for each problem and then a numerical example is developed to demonstrate the applicability of the algorithm.